is obviously the result of libration motions of the body in region $T^2 \times [0, 1] \subset SO(3)$.

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ON THE STABILITY OF PERMANENT ROTATION OF A HEAVY SOLID BODY ABOUT A FIXED POINT

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The permanent rotation of a heavy solid body about its principal axis of inertia with a fixed point is considered. Stability is investigated with the use of the theorem on the stability of Hamiltonian systems with two degrees of freedom in the general elliptic case. It is shown that in the absence of certain resonance relationships in the region of necessary stability conditions, which does not coincide with the region of known sufficient conditions, the first approximation indicates the existence of stability, except possibly, in the case when the parameters of the problem lie on some specific manifolds of the parameter space. Subregions that are free of such exceptional manifolds are indicated in each region of necessary stability conditions.

Necessary stability conditions for permanent rotation about principal axes of inertia of a solid body were investigated by Grammel [3]. Sufficient conditions that matched necessary conditions were obtained by Chetaev in the case of Lagrange integrability [4], and by Rumiantsev in that of Kowalewska integrability [5]. Permanent rotation of a body with arbitrary mass distribution about its principal axis of inertia was considered in [6 — 8], where sufficient stability conditions were established by Chetaev's method and on the basis of the Routh theorem. Bifurcation of permanent rotations and changes of stability were investigated in [9].

1. We introduce the orthogonal system of coordinates $Oxyz$, with the Oz-axis coinciding with the direction of the gravity force mg acting on the body, and also the system of coordinates $OXYZ$ whose axes lie along the principal axes of inertia of the body with respect to the fixed point O. The position of the moving system of coordinates relative to the fixed system $Oxyz$ is specified by the three angles ψ, α and β defining three successive rotations. Angle ψ defines the rotation about the Oz-axis, angle α defines the rotation about the new position of the Ox-axis, and angle β the rotation about the Oy-axis in its new position.

The motion of the solid body will be defined by the variables ψ, α and β and their conjugate canonical momenta P_ψ, P_α and P_β.

The Hamiltonian H expressed in terms of canonical variables is of the form

$$H = \frac{1}{2}\Big[\frac{1}{A\cos^2\alpha}(P_\alpha\cos\alpha\cos\beta + P_\beta\sin\alpha\sin\beta - P_\psi\sin\beta)^2 + \tag{1.1}$$

$$\frac{1}{B}P_\beta^2 + \frac{1}{C\cos^2\alpha}(P_\alpha\cos\alpha\sin\beta - P_\beta\sin\alpha\cos\beta + P_\psi\cos\beta)^2\Big] -$$

$$mg(-x_0\cos\alpha\sin\beta + y_0\sin\alpha + z_0\cos\alpha\cos\beta)$$

$$P_\psi = -Ap\cos\alpha\sin\beta + Bq\sin\alpha + Cr\cos\alpha\cos\beta$$

$$P_\alpha = Ap\cos\beta + Cr\sin\beta, \quad P_\beta = Bq$$

where x_0, y_0, z_0 are the coordinates of the body center of mass in the system of coordinates $OXYZ$, p, q, r are projections of the vector of the body instantaneous angular velocity, defined in the usual manner by ψ, α and β and their derivatives with respect to time, and A, B and C are the principal momenta of inertia of the body relative to the fixed point.

In what follows we consider Hamilton equations for the variables α, β, P_α and P_β with function H (1.1) in which the momentum corresponding to the cyclic coordinate, $P_\psi = l = \text{const}$.

2. Let us investigate the stability of permanent rotation about the principal axis of inertia OZ $(x_0 = y_0 = 0)$.

The unperturbed motion is defined by the following values of variables:

$$\alpha_0 = \beta_0 = \alpha_0^{\cdot} = \beta_0^{\cdot} = 0, \quad \psi_0^{\cdot} = \text{const}$$

$$p_0 = q_0 = 0, \quad r_0 = \psi_0^{\cdot}, \quad P_{\alpha 0} = P_{\beta 0} = 0, \quad P_{\psi 0} = Cr_0 = l$$

In the case of perturbed motion we consider the quantities α, β, P_α and P_β to be small and l to be a fixed number. Then, omitting the additive constant, we represent the Hamiltonian H (1.1) in the form of a series in terms of uniform even powers of the variables α, β, P_α and P_β

$$H = \sum_{m=1}^{\infty} H_{2m} \tag{2.1}$$

where the first two terms are of the form

$$H_2 = \frac{1}{2}\Big[\frac{1}{A}\Big(P_\alpha + \frac{l(A-C)}{C}\beta\Big)^2 + \frac{1}{B}\Big(P_\beta - \frac{lB}{C}\alpha\Big)^2 + a_1\alpha^2 + a_2\beta^2\Big] \tag{2.2}$$

$$H_4 = \frac{1}{2}\Big[\frac{A-C}{AC}\Big(P_\alpha^2\beta^2 - l^2\alpha^2\beta^2 + \frac{1}{3}l^2\beta^4 - 2P_\alpha P_\beta\alpha\beta + lP_\alpha\beta\alpha^2 - \frac{4}{3}lP_\alpha\beta^3 + 2lP_\beta\alpha\beta^2\Big) + \frac{1}{C}\Big(P_\beta^2\alpha^2 + \frac{2}{3}l^2\alpha^4 - \frac{5}{3}lP_\beta\alpha^3\Big)\Big] - \frac{a}{24}(\alpha^4 + 6\alpha^2\beta^2 + \beta^4) \qquad (2.3)$$

$$a_1 = \frac{l^2(C-B)}{C^2} + a, \quad a_2 = \frac{l^2(C-A)}{C^2} + a, \quad a = mgz_0$$

If function H_2 is positive definite, we have stability, hence it is interesting to examine the case when H_2 is an alternative function.

We base the investigation of stability with respect to α, β, P_α and P_β of the zero solution of Hamilton equations with function H defined by (2.1) — (2.3) on the theorem about the stability of Hamiltonian systems with two degrees of freedom [1, 2]. The formulation of the theorem presupposes that function H has been reduced up to and including its fourth order terms by real canonical transformation to the normal form

$$H = \frac{1}{2}\sum_{j=1}^{2}\omega_j r_j + \frac{1}{4}\sum_{j,k=1}^{2}\alpha_{jk}' r_j r_k + \sum_{m=5}^{\infty} H_m \qquad (2.4)$$

$$r_j = \xi_j^2 + \eta_j^2, \quad \alpha_{12}' = \alpha_{21}'$$

where ω_j and α_{jk}' are real constants. The equilibrium position is, according to Liapunov, stable, if the inequality

$$D \equiv \alpha_{11}'\omega_2^2 - 2\alpha_{12}'\omega_1\omega_2 + \alpha_{22}'\omega_1^2 \neq 0 \qquad (2.5)$$

is satisfied.

3. We pass to the determination of the invariants ω_j and α_{jk}' $(j, k = 1, 2)$ of the normal form (2.4). Below besides (2.4) we shall consider for H the following normal form:

$$H = \sum_{j=1}^{2}\varkappa_j u_j v_j + \sum_{j,k=1}^{2}\alpha_{jk}u_j v_j u_k v_k + \sum_{m=5}^{\infty} H_m \qquad (3.1)$$

$$\varkappa_j = I\omega_j, \quad \alpha_{jk} = -\alpha_{jk}'$$

which is derived from (2.4) by the canonical transformation

$$\xi_j = \frac{1}{\sqrt{2}}(u_j + Iv_j), \quad \eta_j = \frac{I}{\sqrt{2}}(u_j - Iv_j)$$

First, we reduce the second order terms H_2 of function H to the normal form. This can be done by the linear simplicial canonical transformation that is constructed with the use of eigenvectors of matrix G which appears in equations and variations

$$dP/dt = GP, \quad P = \{\alpha, \beta, P_\alpha, P_\beta\} \qquad (3.2)$$

$$G = \left\| \begin{matrix} 0 & \frac{A-C}{AC}l & \frac{1}{A} & 0 \\ -\frac{l}{C} & 0 & 0 & \frac{1}{B} \\ -\left(\frac{l^2}{C}+a\right) & 0 & 0 & \frac{l}{C} \\ 0 & \frac{A-C}{AC}l^2 - a & \frac{C-A}{AC}l & 0 \end{matrix} \right\|$$

Let us consider the case when the roots $\varkappa_1$, $-\varkappa_1$, $\varkappa_2$ and $-\varkappa_2$ of the characteristic equation of system (3.2)

$$\varkappa^4 + 2S_1\varkappa^2 + S_2 = 0 \tag{3.3}$$

$$S_1 = \frac{1}{2}\left[\left(1 + \frac{(A-C)(B-C)}{AB}\right)\frac{l^2}{C^2} + \frac{A+B}{AB}a\right], \quad S_2 = \frac{a_1a_2}{AB}$$

are purely imaginary and there are no equal ones among them. For this it is necessary and sufficient for the following concurrent inequalities

$$S_1 > 0, \quad S_2 > 0, \quad S_3 = S_1^2 - S_2 > 0$$

to be satisfied.

Let $\Delta_m(\varkappa)$ $(m = 1, 2, 3, 4)$ be the cofactor taken with the opposite sign of the m-th element of the last row of matrix $G - \varkappa E$ (E is a unit matrix)

$$\Delta_1(\varkappa) = \frac{\varkappa l}{ABC}(A + B - C), \quad \Delta_2(\varkappa) = \frac{1}{B}\left(\varkappa^2 + \frac{a_1}{A}\right) \tag{3.5}$$

$$\Delta_3(\varkappa) = \frac{l}{C}\left(\varkappa^2 + \frac{C-A}{AB}a_1\right), \quad \Delta_4(\varkappa) = \varkappa\left(\varkappa^2 + \frac{l^2}{C^2} + \frac{a}{A}\right)$$

The sought transformation is then

$$P = \Lambda Q, \quad Q = \{x_1, \ x_2, \ y_1, \ y_2\} \tag{3.6}$$

$$\Lambda = \left\| \begin{matrix} \dfrac{\Delta_1(\varkappa_1)}{I\delta_1} & \dfrac{\Delta_1(\varkappa_2)}{I\delta_2} & -\dfrac{\Delta_1(\varkappa_1)}{\delta_1} & -\dfrac{\Delta_1(\varkappa_2)}{\delta_2} \\ \dfrac{\Delta_2(\varkappa_1)}{I\delta_1} & \dfrac{\Delta_2(\varkappa_2)}{I\delta_2} & \dfrac{\Delta_2(\varkappa_1)}{\delta_1} & \dfrac{\Delta_2(\varkappa_2)}{\delta_2} \\ \dfrac{\Delta_3(\varkappa_1)}{I\delta_1} & \dfrac{\Delta_3(\varkappa_2)}{I\delta_2} & \dfrac{\Delta_3(\varkappa_1)}{\delta_1} & \dfrac{\Delta_3(\varkappa_2)}{\delta_2} \\ \dfrac{\Delta_4(\varkappa_1)}{I\delta_1} & \dfrac{\Delta_4(\varkappa_2)}{I\delta_2} & -\dfrac{\Delta_4(\varkappa_1)}{\delta_1} & -\dfrac{\Delta_4(\varkappa_2)}{\delta_2} \end{matrix} \right\|$$

with

$$I\delta_j^2 = 2\left(\Delta_1(\varkappa_j)\Delta_3(\varkappa_j) - \Delta_4(\varkappa_j)\Delta_2(\varkappa_j)\right), \qquad j = 1, 2 \tag{3.7}$$

Transformation (3.6) reduces matrix G to the diagonal form and the quadratic terms of function H to the normal form. This transformation is canonical and universal, since by virtue of (3.6) the expression

$$P_\alpha d\alpha + P_\beta d\beta - y_1dx_1 - y_2dx_2$$

is a total differential. This can be readily verified by taking into account that the constants $\Delta_m(\varkappa_j)$ (3.5) satisfy the relationships

$$\Delta_1(\varkappa_1)\Delta_3(\varkappa_2) - \Delta_2(\varkappa_2)\Delta_4(\varkappa_1) = 0$$

$$\Delta_1(\varkappa_2)\Delta_3(\varkappa_1) - \Delta_2(\varkappa_1)\Delta_4(\varkappa_2) = 0$$

The transformation (3.6) is complex-valued, as it must be, if together with the transformation

$$\xi_j = \frac{1}{\sqrt{2}}(x_j + Iy_j), \quad \eta_j = \frac{I}{\sqrt{2}}(x_j - Iy_j) \tag{3.8}$$

which reduces Hamiltonian H to the real form (2.4), it is to yield a real-valued canonical transformation. Since the constants $\Delta_1(\varkappa_j)$ and $\Delta_4(\varkappa_j)$ are purely imaginary, while $\Delta_2(\varkappa_j)$ and $\Delta_3(\varkappa_j)$ are real, the transformation α, β, P_α, $P_\beta \to \xi_1$, ξ_2, η_1, η_2 (3.6), (3.8) is real when the constants δ_1 and δ_2 are real.

Formula (3.7) for $I\delta_j^2$ can be represented as follows:

$$I\delta_j^2 = I\omega_j\,[\sigma_1 + (-1)^j\sigma_2] \tag{3.9}$$

where the real numbers σ_1 and σ_2 are independent of j. It is always possible to obtain on the basis of (3.9) $\delta_j^2 > 0$ by the selection of signs of $\omega_j = -I\varkappa_j$. Hence the real valuedness of transformation (3.6), (3.8) determines whether the frequencies ω_j in the normal form (2.4) are of the same or different sign. The interesting case when $\omega_1\omega_2 < 0$ is characterized by that the signature of the quadratic form of H_2 is zero

$$a_1 < 0, \quad a_2 < 0 \tag{3.10}$$

When $\delta_1\delta_2 \neq 0$, the transformation (3.6) is nonsingular, which yields the condition

$$\delta_1^2\delta_2^2 \equiv 4\omega_1\omega_2 \frac{l^2(A+B-C)^2}{A^2B^3C^2} S_3 a_1 \neq 0 \tag{3.11}$$

Since by assumption inequalities (3.4) are satisfied, hence $\omega_1\omega_2 a_1 S_3 \neq 0$. The stipulation that condition (3.11) must be satisfied makes it necessary to exclude from our considerations the case of disk $A + B - C = 0$. However in that case

$$\varkappa_1^2 = -\frac{l^2}{C^2} - \frac{a}{A}, \quad \varkappa_2^2 = -\frac{l^2}{C^2} - \frac{a}{B}$$

which means that $a_1 > 0$, $a_2 > 0$ and function H_2 is positively determined.

Grammel [3] had carried out a similar analysis of necessary conditions of stability (3.4) on the basis of which it is possible to separate in the first approximation the following stability regions that satisfy inequalities (3.10):

$$z_0 > 0, \quad C < B \leqslant A, \quad l > l_2 \tag{3.12}$$

$$z_0 < 0, \quad C \geqslant A \geqslant B, \quad R > 0, \quad l_3 < l < l_2 \tag{3.13}$$

$$z_0 < 0, \quad A \geqslant C \geqslant B, \quad R > 0, \quad l_3 < l < l_2 \tag{3.14}$$

$$z_0 < 0, \quad C < B \leqslant A, \quad l > l_3 \tag{3.15}$$

where

$$R = C^2 + B^2 + 3AB - 2C(A + B)$$

$$l_2^2 = \frac{aC^2}{B-C}, \quad l_3^2 = \frac{4AB - AC - BC + 2\sqrt{AB(2A-C)(2B-C)}}{C-A-B}\,a$$

4. To reduce the fourth power terms of function H in (2.3) we use the Birkhoff transformation [10], and introduce the canonical variables u_j and v_j $(j = 1, 2)$ by formulas

$$u_j = \frac{\partial K}{\partial v_j}, \quad y_j = \frac{\partial K}{\partial x_j}, \quad K = \sum_{j=1}^{2} v_j x_j + K_4 \tag{4.1}$$

where K_4 is a homogeneous fourth power polynomial of the variables v_j and x_j.

In the absence of resonance of the form

$$n\omega_1 + m\omega_2 = 0 \tag{4.2}$$

where n and m are integers that satisfy the equality $|n| + |m| = 4$, the constant coefficients of polynomial K_4 may be chosen so that in the new variables u_j and v_j all terms which do not appear in the normal form can be excluded from function H_4 and, consequently, the Hamiltonian H will be of the form (3.1).

Below we assume that equality (4.2) does not apply, and consider the general (nonresonance) case.

Applying transformations (3.6) and (4.1) to function H_4, for the invariants α_{11}, α_{22} and α_{12} of normal form, we obtain the following formulas:

$$\alpha_{jj} = \Psi_1(\varkappa_j)\,\delta_j^{-4}, \qquad j=1,2 \tag{4.3}$$

$$\alpha_{12} = (\Psi_2(\varkappa_1, \varkappa_2) + \Psi_2(\varkappa_2, \varkappa_1) + \frac{a}{2}\Psi_3(\varkappa_1)\Psi_3(\varkappa_2))\,\delta_1^{-2}\delta_2^{-2}$$

$$\Psi_1(x) = \frac{C-A}{AC}\Delta_2(x)[\Delta_2(x)(3\Delta_3^2(x) + l^2\Delta_1^2(x) + l^2\Delta_2^2(x) - 4l\Delta_2(x)\Delta_3(x) - 2l\Delta_1(x)\Delta_4(x)) + \Delta_1(x)(2\Delta_3(x)\Delta_4(x) - l\Delta_1(x)\Delta_3(x))] - \frac{1}{C}\Delta_1^2(x)(3\Delta_4^2(x) + 2l^2\Delta_1^2(x) - 5l\Delta_1(x)\Delta_4(x)) + \frac{a}{4}(\Delta_1^2(x) - \Delta_2^2(x))^2$$

$$\Psi_2(x,y) = \frac{C-A}{AC}[\Delta_2^2(y)(\Delta_3^2(x) + l^2\Delta_2^2(x) + l^2\Delta_1^2(x)) + \Delta_1(x)\Delta_2(y)(2\Delta_3(y)\Delta_4(x) - l\Delta_1(x)\Delta_3(y) - 2l\Delta_2(y)\Delta_4(x)) + 2\Delta_2(x)\Delta_2(y)\Delta_3(y)(\Delta_3(x) - 2l\Delta_2(x))] - \frac{\Delta_1(x)}{C}[2\Delta_1(y)(\Delta_4(x)\Delta_4(y) + l^2\Delta_1(x)\Delta_1(y)) + \Delta_1(x)\Delta_4(y)(\Delta_4(y) - 5l\Delta_1(y))]$$

$$\Psi_3(x) = \Delta_1^2(x) - \Delta_2^2(x)$$

The analysis of inequality (2.5) in its general form is difficult, it is, however, possible to show that in the regions (3.12) — (3.15) D is an analytic function of l which does not identically vanish, and that in each of these four regions exists a subregion in which $D \neq 0$ (*). Note that the case of $A = B$ investigated in [4] may be excluded from our analysis.

5. Let us, first, consider regions (3.12) and (3.15) and examine the quantity D as a function of parameter $\mu = C/l$ in some ring $\mu_1 < |\mu| < \mu_2$ in the complex plane μ, where $\mu_2 > \mu_1 > 0$ and μ_1 can be arbitrarily small. We shall show that in that ring D is an analytic function of μ and determine the principal part of expansion of this function in a Laurent series in powers of parameter μ. The roots $\varkappa_1^2$ and $\varkappa_2^2$ of the secular equation (3.3) can be represented in the form of series in powers of parameter μ^2

$$\varkappa_1^2 = -\frac{1}{\mu^2} + \chi_1(\mu^2), \quad \varkappa_2^2 = -\frac{(A-C)(B-C)}{\mu^2 AB} + \chi_2(\mu^2) \tag{5.1}$$

where $\chi_1(\mu^2)$ and $\chi_2(\mu^2)$ are the Taylor parts of related expansions. The value $\mu_3^2 = C^2/l_3^2$, which is the root of equation $S_3 = 0$, corresponds to the singular point of functions $\varkappa_1^2$ and $\varkappa_2^2$ closest to $\mu = 0$, hence series $\chi_1(\mu^2)$ and $\chi_2(\mu^2)$ converge in the circle

$$0 \leqslant |\mu| < |\mu_3| \tag{5.2}$$

For $\Delta_m(\varkappa_j)$ ($m = 1, 2, 3, 4$; $j = 1, 2$) in (3.5) we have

*) It is shown in the recently published paper [11] that $D \not\equiv 0$.

$$\Delta_1(\varkappa_j) = \frac{\varkappa_j}{\mu AB}(A+B-C), \quad \Delta_2(\varkappa_1) = \frac{C-A-B}{\mu^2 AB} + \chi_{21}(\mu^2) \quad (5.3)$$

$$\Delta_2(\varkappa_2) = \frac{C-A-B}{\mu^3 AB^2}(B-C) + \chi_{22}(\mu^2)$$

$$\Delta_3(\varkappa_1) = \frac{1}{\mu}\left[\frac{(C-A-B)C}{\mu^2 AB} + \chi_{31}(\mu^2)\right]$$

$$\Delta_3(\varkappa_2) = \frac{1}{\mu}\left[\frac{a(2A-C)}{ABC}(B-C) + \chi_{32}(\mu^2)\right]$$

$$\Delta_4(\varkappa_1) = \varkappa_1\left[\frac{a(C-2A)}{AC} + \chi_{41}(\mu^2)\right]$$

$$\Delta_4(\varkappa_2) = \varkappa_2\left[\frac{(A+B-C)C}{\mu^2 AB} + \chi_{42}(\mu^2)\right]$$

where $\chi_{kj}(\mu^2)$ $(k = 2, 3, 4;\ j = 1, 2)$ are series in positive integral powers of parameter μ^2. The expansions of functions $\chi_{kj}(\mu^2)$ converge in zero in the circle (5.2). Representing $\delta_1{}^2$ and $\delta_2{}^2$ (3.7) by Laurent series in the ring $\mu_1 < |\mu| < \mu_3$, we obtain expansions in which we separate the first terms

$$\delta_1{}^2 = \frac{\omega_1}{\mu^4}\left[-\frac{2C(A+B-C)^2}{A^2B^2} + \chi_3(\mu^2)\right] \quad (5.4)$$

$$\delta_2{}^2 = \frac{\omega_2}{\mu^4}\left[\frac{2C(B-C)(A+B-C)^2}{A^2B^3} + \chi_4(\mu^2)\right]$$

where functions $\chi_3(\mu^2)$ and $\chi_4(\mu^2)$ are of the same form as functions $\chi_{kj}(\mu^2)$, and $\chi_3(0) = \chi_4(0) = 0$.

Using formulas (5.1), (5.3), (5.4), (4.3) and (2.5), we represent D by a Laurent series in powers of parameter μ^2

$$D = \frac{2C-A-B}{4AB\mu^2} + \chi(\mu^2) \quad (5.5)$$

which converges in the ring $\mu_1 \leqslant |\mu| < \mu^*$ with $\mu^* = \min(C/|l_2|, |\mu_3|)$, and where $\chi(\mu^2)$ is the Taylor part of the expansion. As a function of parameter $l = C/\mu$, series (5.5) is convergent for all finite l lying outside the circle $|l| = |l_3|$. In the case of (3.12) all roots of equation (in l) $\delta_1{}^2\delta_2{}^2 = 0$ which satisfies inequality (3.11) and, consequently, all singular points of the finite plane for function D lie either on the imaginary axis, or on the real axis to the left of the singular point $l_2 > 0$. Hence in that case the expansion of function $\chi(\mu)$ can be analytically continued along the real axis of the complex plane l up to $l = l_2$. In the case of (3.15) all singular points of function D similarly lie either along the imaginary axis or on the real axis to the left of point $l_3 > 0$ and, consequently, the series $\chi(\mu)$ can be analytically extended along the real axis up to point l_3.

Thus the quantity D is an analytic function for all finite values of $l > 0$ admissible in the case of (3.12) and (3.15). In the considered cases $A > B > C$, hence for fairly great l the quantity D is nonzero.

6. Let us consider regions (3.13) and (3.14) on the assumption that $B \neq C$. Note that for $B = C \neq A$, $l_2 = +\infty$, and we have the case considered in Sect. 5. We shall show that in the interval $l_3 < l < l_2$ the quantity D is an analytic function of l and that in a reasonably small neighborhood of l_2, $D \neq 0$. We introduce the small parameter $\lambda > 0$ and assume that

$$l^2 = l_2^2 - \lambda^2 C^2 \tag{6.1}$$

then in the circle $|\lambda^2| < (l_2^2 - l_3^2)/C^2$ we have for $\varkappa_1^2$ and $\varkappa_2^2$ expansions in positive integral powers of λ^2 with

$$\varkappa_1^2 = \lambda^2\left[-\frac{(C-B)(A-B)}{R} + \varphi_1(\lambda^2)\right], \quad \varkappa_2^2 = -\frac{Rl_2^2}{ABC^2} + \varphi_2(\lambda^2) \tag{6.2}$$

where the series $\varphi_j(\lambda^2)$ are such that $\varphi_1(0) = \varphi_2(0) = 0$. Using (3.5) and (6.1), for $\Delta_m(\varkappa_j)$ we obtain

$$\Delta_1(\varkappa_j) = \frac{\varkappa_j l}{ABC}(A + B - C), \quad \Delta_2(\varkappa_j) = \frac{1}{B}\left(\varkappa_j^2 - \frac{C-B}{A}\lambda^2\right) \tag{6.3}$$

$$\Delta_3(\varkappa_j) = \frac{l}{C}\left(\varkappa_j^2 - \frac{(A-C)(B-C)}{AB}\lambda^2\right)$$

$$\Delta_4(\varkappa_j) = \varkappa_j\left(\frac{A+B-C}{AC^2}\, l_2^2 + \varkappa_j^2 - \lambda^2\right)$$

We represent the constants δ_j^2 in the form

$$\delta_1^2 = \omega_1\lambda^2\left[\frac{2l_2^2(C-B)}{A^2B^2C^2}(A+B-C)^2 + \varphi_3(\lambda^2)\right] \tag{6.4}$$

$$\delta_2^2 = \omega_2\left(-\frac{2l_2^4}{A^2B^3C^4}R^2 + \varphi_4(\lambda^2)\right)$$

where functions $\varphi_3(\lambda^2)$ and $\varphi_4(\lambda^2)$ are of the same kind as functions $\varphi_j(\lambda^2)$ $(j = 1, 2)$. Now, taking into consideration formulas (2.5), (4.3) and (6.1) — (6.4), for function D in the ring

$$\lambda_1 \leqslant |\lambda^2| < (l_2^2 - l_3^2)/C^2$$

where $\lambda_1 > 0$ is an arbitrarily small number, we have the Laurent series in powers of parameter λ^2

$$D = \frac{3l_2^4(3C-4B)(B-A)}{16ABC^5\lambda^2} + \varphi(\lambda^2) \tag{6.5}$$

where $\varphi(\lambda^2)$ is the Taylor part of the expansion. When the condition $A > B$ and $C > B$ is satisfied, the quantity D in (6.5) can vanish at small λ only for $3C - 4B = 0$, and it is then possible to show that the first term of series (6.5) is

$$\varphi(0) = \frac{l_2^2(3C-4A)}{256AC^2R}(52A - 11C)$$

which for $A > B$ is nonzero.

7. In the considered regions (3.12) — (3.15) the quantity D is an analytic function of l and for $l \gg 1$, $D \neq 0$ hence only a finite number of values of l exists for every fixed set of constants A, B, C and z_0 for which $D = 0$.

Let us formulate the obtained result.

Theorem. If the frequencies ω_1 and ω_2 are different and have no fourth order resonances, the necessary stability conditions in the first approximation (2.6) for permanent rotation of a heavy solid body about its principal axis of inertia are also the sufficient conditions, perhaps with the exception of those values of parameters A, B, C, z_0 and l for which D (2.5) vanishes. In regions (3.12) — (3.15) the equation $D = 0$ in l has only a finite number of roots, and each of these regions contains a subregion where $D \neq 0$.

Stability of permanent rotation with respect to the variables α, β, P_α and P_β for $D \neq 0$ has been proved on the assumption that the constant l (or ψ_0) is not subjected to perturbations. If l is imparted a fairly small increment Δl, the total perturbed motion can be represented as that of some other permanent rotation for fixed $l_1 = l + \Delta l$. The set of permanent rotations is continuous with respect to l, and each permanent rotation in the fairly small neighborhood of the considered one is conditionally stable. It is then possible to make the statement about the absolute stability with respect to the variables α, β, P_α and P_β, as was made in the observation in [12] about the character of stability established on the basis of the Routh's theorem.

When the indicated above conditions are satisfied, permanent rotations are also stable with respect to the variables p, q, r, γ, γ' and γ''.

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